A New Method for The Evaluation of Pair Error Probabilities in the Correlated Rayleigh Fading Channel

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Abstract—**We present an new method for the analytical calculation of the error probability in a Rayleigh fading channel with correlated fading amplitudes. Even though the method can be applied for more general problems, we restrict ourselves on the investigation of a multicarrier CDMA transmission. The frequency correlation between the fading amplitudes of the carriers is given by the delay power spectrum of the channel. We generalize the well-known analytical expressions for the pairwise error probability of the uncorrelated Rayleigh channel to this case of correlated fading. This probability can be expressed in a closed form by an integral over a finite interval which can be easily evaluated numerically. We present some examples to show how much of the diversity of a code remains useful if the transmission channel does not provide an arbitrary degree of diversity.**

Keywords— **Multicarrier CDMA, Diversity, Coding, Correlated Rayleigh Fading.**

I. Introduction

RELIABLE transmission over a mobile Rayleigh fading channel needs to make use of some statistical independence that must be provided by the channel. The simplest method is to receive the same symbol over different antennas, different frequencies, different time slots, or different echo paths (for a RAKE receiver). This familiar form of diversity (we will call it repetition diversity because it is nothing but a repetition code) leads to a power law $P_b \propto (E_b/N_0)^{-L}$ for L *independent* diversity branches. The same formula for L-fold repetition diversity with maximum ratio combining (MRC) applies for the pairwise error probability for maximum likelihood (ML) decoding of an error-correcting code, if L is the Hamming distance between the two code words and ideal interleaving can be assumed. Analytical formulas for this error probabilities in a Rayleigh fading channel with independent fading amplitudes are well known [1], [2]. However, for many relevant mobile transmission scenarios, the assumption of independent fading is too optimistic. For example, the interleaving depth may be restricted by the allowed decoding delay or the spreading bandwidth is not very large compared to the correlation bandwidth.

In this paper, we present a new method to calculate this error probabilty for the case that the L diversity branches are *correlated*. First, we make use of the fact that the correlated fading process is unitary equivalent to an uncorrelated fading process. Practically this means that we have solve the eigenvalue problem of the autocorrelation matrix. Then we use the alternative form of the Gaussian probability integral for which the averaging over independent fading amplitudes of different power can easily be performed. To calculate the error rate, one has then to solve a finite integral numerically.

An interesting application where the question of channel diversity becomes important is multicarrier (MC) CDMA [3], [4]. In such a system, typically an outer repetition (RP) code or a Walsh-Hadamard (WH) code is used for spreading. Optionally, an inner convolutional code can be used. The performance of different channel coding schemes for MC-CDMA in a Rayleigh fading channel with independent fading amplitudes has been studied analytically in [5]. We carry out a similar performance analysis only for the outer (RP or WH) coding, but for the correlated Rayleigh fading channel. This paper is organized as follows: We describe the transmission system in Section 2. In Section 3 we derive the formula to calculate the error probability for correlated fading. We then evaluate this formula numerically for the MC-CDMA system in Section 4, and we draw some conclusions in Section 6.

II. System Model

We consider a multicarrier CDMA system with BPSK modulation [3], [4], [5]. One way to introduce spreading is to transmit a pseudo-noise (PN) sequence with plus or minus sign for each data bit. In fact, this corresponds to a simple repetition code and BPSK modulation followed by multiplication with the PN sequence for user separation. Another way is to use low-rate Walsh-Hadamard (WH) codes with BPSK followed by multiplication with the PN sequence. As it will be explained later, it may be advantageous to apply a pseudo random permutation to the WH code word that we regard as an *intra code word interleaver*. Let K be the number of sub-carriers. We assume that an integer number of code words of length N will be transmitted in one time slot, i.e. K is an integer multiple of N. We assume that the channel is frequency selective, but not time selective (slowly varying). Intersymbol interference is assumed to be absorbed e.g. by a guard interval. We may now work with a frequency discrete channel model.

For a fair comparison between different coding schemes, it is a good practice to regard error rates as as function of E_b/N_0, where E_b is the energy per data bit and N_0 is the one-sided white Gaussian noise density. For BPSK and a code of rate R_c we have $E_S = R_c E_b$, where E_S is the energy per BPSK symbol (i.e. the *chirp energy*).

A code word will be mapped on a sequence of BPSK symbols $x_i \in \{\pm\sqrt{E_S}\}, i \in \{1, ..., N\}$. The vector $\mathbf{x} = (x_1, ..., x_N)^{\mathrm{T}}$ corresponding to this code word will be transmitted over the discrete channel given by

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{n}. \tag{1}$$

$\mathbf{y}$ is the vector of received symbols and $\mathbf{n}$ is the complex Gaussian noise vector with variance $\sigma^2 = N_0/2$ in each real component. The fading is decribed by the diagonal matrix $\mathbf{C}$. The diagonal is the vector of fading amplitudes $(c_1, ..., c_N)^{\mathrm{T}}$. All vectors indices should be understood as frequency indices. The fading amplitudes are normalized to average power one. We assume Rayleigh fading and the WSSUS (wide-sense stationary uncorrelated scattering) model, see e.g. [2]. This means that the fading is a complex-valued Gaussian process with mean zero, and the 2-D autocorrelation depends only on time and frequency differences. In this model, the statistics of the fading amplitudes c_i sampled at frequencies f_i is completely determinded by the elements

$$\mathrm{E}\{c_i c_k^*\} = K(f_i - f_k). \tag{2}$$

of the (Hermitian) autocorrelation matrix. $K(f)$ denotes the frequency autocorrelation function of the WSSUS model.

III. Analytical Evaluation of the Pairwise Error Probability

All code words are assumed to be sent with the same *a priori* probability. Then, for optimum (ML or MRC) detection, the pairwise error probability that a vector $\mathbf{x}$ will be send but another vector $\mathbf{x}'$ will be detected under the condition of a fixed channel $\mathbf{C}$ is given by

$$P(\mathbf{x} \to \mathbf{x}' | \mathbf{C}) = \frac{1}{2} \mathrm{erfc}\left(\sqrt{\sum_{l=1}^{L} |c_{i_l}|^2 \frac{E_S}{N_0}}\right), \tag{3}$$

if the two code words differ at exactly $L \leq N$ positions $i_1, ..., i_L$. The pairwise error probability $P_L(\mathbf{x} \to \mathbf{x}')$ for two code word that differ at L positions will be obtained by averaging over the fading amplitudes. If they are independent and identically Rayleigh distributed, the sum of squared amplitudes can be shown to be χ^2 distributed and the average can be performed by evaluating one integral, leading to the well known formula for diversity with L independent branches, see [1], [2].

Recently, it has been pointed out by Simon and Divsalar [6] that the (solved) problem of calculating the error rate for diversity with L independent branches as well as some others (unsolved) problems can be treated in a very elegant way by using the polar representation of the Gaussian probability integral given by

$$\frac{1}{2}\mathrm{erfc}(x) = \frac{1}{\pi}\int_0^{\pi/2} \exp\left(-\frac{x^2}{\sin^2\theta}\right) d\theta. \tag{4}$$

With this formula, the averaging over the L Rayleigh distributed fading amplitudes $c_{i_1}, ..., c_{i_L}$ in equation (3) can be easily performed for independent Rayleigh fading since the exponential factorizes.

We can now use this method to treat also the case of correlated Rayleigh fading by transforming the correlated fading channel to an uncorrelated one. We note that since the autocorrelation matrix of the fading is Hermitian, it can be transformed to a diagonal one by a unitary matrix $\mathbf{U}$. This is just the well known Karhunen-Loeve transform, see e.g. [7]. By this unitary transform, a channel given by a vector $\mathbf{c}_L = (c_{i_1}, ..., c_{i_L})^{\mathrm{T}}$ will be mapped to a vector $\mathbf{b}_L = (b_1, ..., b_L)^{\mathrm{T}}$ of uncorrelated fading amplitudes

$$\mathrm{E}\{b_i b_k^*\} = \lambda_i \delta_{ik} \tag{5}$$

according to

$$\mathbf{b}_L = \mathbf{U^{-1}} \mathbf{c}_L. \tag{6}$$

The real numbers λ_i are just the eigenvalues of the complex autocorrelation matrix of the fading. Without loss of generality, we assume ordering $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_L \geq 0$. Note that the unitary matrix $\mathbf{U}$ and therefore also the eigenvalues λ_i depend on the L positions $i_1, ..., i_L$. Since the transform is unitary, it follows that

$$\sum_{i=1}^{L} \lambda_i = \mathrm{E}\{\sum_{i=1}^{L} |b_i|^2\} = \mathrm{E}\{\sum_{l=1}^{L} |c_{i_l}|^2\} = L \qquad (7)$$

We can interpret λ_i as the power of the $i-$th diversity branch of the *equivalent independently fading channel* obtained by a unitary transform.

The averaging over L in equation (3) can now easily performed, since the exponential factorizes, leading to the expression:

$$P_L(\mathbf{x} \to \mathbf{x}') = \frac{1}{\pi} \int_0^{\pi/2} \prod_{i=1}^{L} \frac{1}{1 + \frac{\lambda_i}{\sin^2 \theta} \frac{E_S}{N_0}} d\theta. \qquad (8)$$

Again, this integral can be evaluated numerically.

IV. Numerical Analysis

A. Repetition Diversity

For the L -fold repetition code, there are only two code words and the pairwise error probability equals the bit error probability P_b given by eq. (8) with $E_S = E_b/L$. The decay depends on the diversity branch spectrum $\{\lambda_i\}_{i=1}^{L}$ of the eigenvalues. For the numerical investigations, we will assume an exponential delay power spectrum with corresponding frequency autocorrelation function is given by

$$K(f) = \frac{1}{1 + \mathrm{j}2\pi f \tau_m}. \qquad (9)$$

Fig. 1 shows the first 16 eigenvalues for a spreading factor $L = 64$ and different values of the bandwidth B. We define a *normalized* bandwidth $X = B\tau_m$. We have assumed that the BPSK symbols are equally frequency-spaced over the bandwidth. We see that for a small bandwidth (e.g. $X = 1$ corresponding to 1 MHz for $\tau = 1\mu s$), the equivalent independent fading channel has only a low number of diversity branches with significant power. We found that the *diversity branch spectrum* like shown in Fig. 1 is nearly independent of L if L is significantly greater than X. It is therefore a very useful quantity to characterize the diversity that can be provided by the channel.

A look at the eigenvalues gives a first glance how many diversity branches of the equivalent independent fading channel contribute significantly to the transmission. It finds it's reflection in the performance curves. Fig. 2 shows the pairwise (=bit) error probability for $L = 32$ and $X = B\tau_m = 0.5, 1, 2, 4, 8, 16$. The high diversity degree of the repetition code ($L = 32$) can show a high diversity gain if the equivalent channel has enough independent diversity branches of significant power. This is the case for e.g. $X = 16$, but not for $X = 1$ or $X = 2$. For low X a lower repetition rate L would have been sufficient.

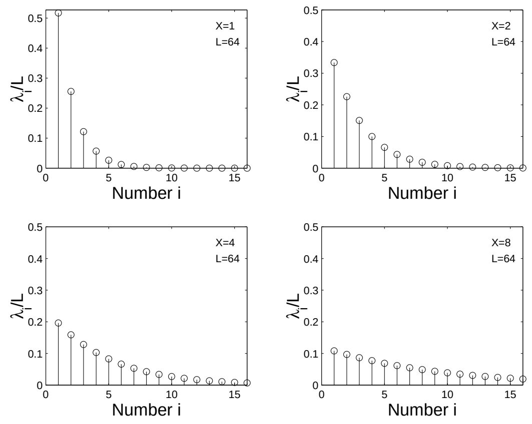

Fig. 1. Diversity branches of the equivalent independent fading channel and normalized bandwidth $X = B\tau_m = 1, 2, 4, 8$.

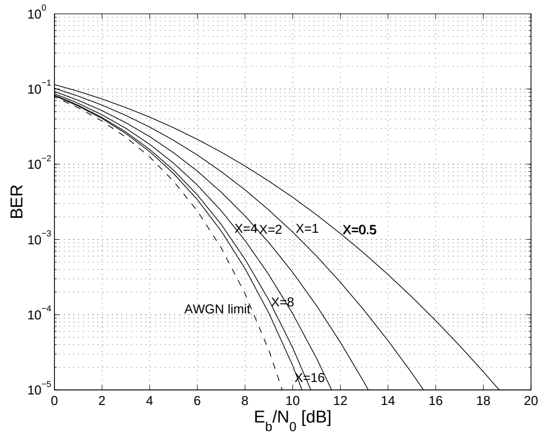

Fig. 2. Bit error probabilities for 32-fold repetition diversity with $X = 0.5, 1, 2, 4, 8, 16$

B. Walsh-Hadamard Codes

We now consider a $\mathrm{WH}(M, \log_2 M, M/2)$ code of M different code words, code rate $R_c = \log_2 M/M$, and constant weight $M/2$, which means that each pair of code words differs in $L = M/2$ symbol positions. The pairwise error probability given by eq. (8) with $E_S = \frac{\log_2 M}{M} E_b$ is different for different pairs $\mathbf{x}$ and $\mathbf{x}'$ and depends on the L positions $i_1, ..., i_L$ where the two code words differ. The autocorrelation matrix and therefore the eigenvalues λ_i depend on these positions.

The correlations of the channel sampled at these positions can be very different. Assume, for example, that the all-zero code word has been send, which corresponds to the vector $\mathbf{x_1}$ with the symbol $+\sqrt{E_S}$ in each position. If $\mathbf{x}'$ is the vector with symbol $+\sqrt{E_S}$ in position $i = 1, 2, .., M/2$ and $-\sqrt{E_S}$ in position $i = M/2+1, ..., M$, there is much more correlation of the fading amplitudes than for $\mathbf{x}'$ being the vector with alternating sign (i.e. $x'_i = (-1)^{i-1}\sqrt{E_S}$). Numerical evaluation of the pairwise error probabilities shows that the curves for different pairs can differ by several dezibels. We found that this effect - that is unknown for uncorrelated fading - can be mitigated significantly if we introduce an *intra code word interleaving* which is just a pseudo random permutation of the coded symbols.

For practical considerations, the bit error probability P_b is much more interesting than the pairwise error probability. The union bound for the bit error probability is given by

$$P_b \leq \frac{M/2}{M-1} \sum_{m=2}^{M} P_{M/2}(\mathbf{x_1} \rightarrow \mathbf{x_m}). \qquad (10)$$

Fig. 3 shows the union bound for the bit error rate for $M = 64$ and different values of the normalized bandwidth X. The WH(64,6,32) code has nearly the

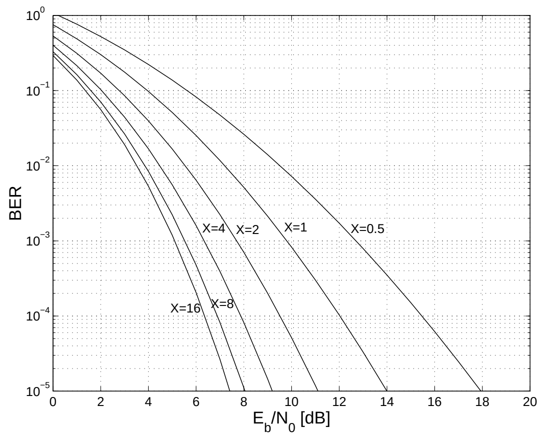

Fig. 3. Bit error rates (Union Bounds) for correlated Rayleigh fading ($X = 0.5, 1, 2, 4, 8, 16$) and the WH code with $M = 64$ and intra code word interleaving.

same spreading factor R_c^{-1} like the $L = 10$ repetition code, but the performance is much better. One reason is the high diversity degree that is given by the Hamming distance $d_H = 32$. The other reason is that we have a coding gain $G = \lg(d_H R_c)\,\text{dB} > 0\,\text{dB}$. Therefore, this code even outperforms the $L = 32$ repetition code with the same Hamming distance. For high values of X, the performance is significantly better than the AWGN limit. We conclude from this figure that a high Hamming distance of a code only leads to a steep decay of the performance curves if the equivalent channel has enough independent diversity branches, i.e. X must be large enough. For the strong WH code with $M = 64$, Fig. 3 shows that there is a loss of approx. 10 dB (at $BER = 10^{-4}$) between the most frequency selective channel with $X = 16$ and the very flat one with $X = 0.5$. Since a system in a mobile environment must operate in very different channels, the system design must take care of these degradations.

V. Conclusions

We have presented a method for the analytical evaluation of the pairwise error probability for coded transmission in a correlated Rayleigh fading channel. It is given in a closed form as an integral over a finite interval which can be evaluated numerically. The relevant quantities are the eigenvalues λ_i of the frequency autocorrelation matrix. They can be regarded as the branch power distribution of an *equivalent independently fading diversity channel.* Even though applied only for the special case of a MC-CDMA system, the method can be easily applied to similar problems where correlated fading occurs, e.g. coded transmission with insufficient time and/or frequency interleaving, frequency hopping, and other related topics.

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