Geometrical Properties of Orthogonal Space-Time Codes

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Abstract— **In this letter, we discuss the geometrical properties of a transmission scheme with orthogonal space-time codes. In particular, we show that the transmission channel can be interpreted as the rotation of a vector of transmit symbols in a Euclidean space, together with an attenuation and additive noise. We show that the receiver - as intuitively obvious - essentially has to perform a back- rotation. This geometrical interpretation applies to real vector spaces of signals, i.e., the complex transmit and receive symbols have to be split up into their real and imaginary parts.**

Keywords— **Space-time coding, diversity, fading channels, wireless communications.**

I. Introduction

DURING the last few years, multiple antenna transmission evolved into a wide-spread area of research. One reason for this is the fact that it offers the possibility to transfer complexity from the receiver to the transmitter. Especially the two-antenna transmission setup that had been ingeniously handcrafted by Alamouti [1] inspired a series of work on so-called space-time block codes [2]. The Alamouti scheme provides a diversity degree of two with no loss in data rate. At the receiver site, a simple combiner is able to disentangle the superposed transmit symbols. This concept was further developed and put into a general theoretical framework by Tarokh et al. [3], who noticed that the desirable properties were related to the orthogonal design of the code matrix. They classified the orthogonal design constructions and showed that it is not possible to extend the Alamouti scheme to more antennas without loss in bandwidth efficiency. They found constructions for *generalized* orthogonal designs with non-square code matrices for any number of antennas, but with only half the maximal bandwidth efficiency and a delay growing exponentially with the number of antennas. Tirkkonen and Hottinen [6] proved that complex orthogonal designs with square code matrices exist only for a number of transmit antennas that is a power of two. These codes have minimal delay, but their bandwidth efficiency decreases exponentially.

The common property of all these space-time code designs is a code matrix with orthogonal colums that leads to a simple linear processing at the receiver site. The goal of this letter is to provide an intuitively simple understanding of these properties by visualizing the transmission setup by geometrical concepts. It turns out that the transmission channels itself acts as kind of rotation in a real Euclidean vector space. As a consequence, the familiar complex vector space notation has to be left by splitting up the transmit and receive symbols into their respective real and imaginary parts. This is due to the fact that the code matrix is not linear in the complex modulation symbols, but it is linear in the real and imaginary parts of them.

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II. Definitions and Abbreviations

Let $\mathcal{R}^N$ and $\mathcal{C}^N$ denote the N-dimensional real and complex space, respectively. For each complex vector $\mathbf{z} = (z_1, ..., z_N)^{\mathrm{T}} \in \mathcal{C}^N$ there is a corresponding real vector $\mathbf{x} = (x_1, ..., x_{2N})^{\mathrm{T}} \in \mathcal{R}^{2N}$ given by

$$\mathbf{x} = \begin{pmatrix} \Re\{\mathbf{z}\} \\ \Im\{\mathbf{z}\} \end{pmatrix}. \tag{1}$$

We will write $\mathbf{z} \leftrightarrow \mathbf{x}$ for this correspondence. Both vectors have the same length $\|\mathbf{x}\| = \|\mathbf{z}\|$ (the norm in their respective vector spaces). The vector of the first N components of $\mathbf{x}$ will sometimes be denoted by $\mathbf{x}'$, and the vector of the last ones by $\mathbf{x}''$. We write $\mathbf{x} \cdot \mathbf{y}$ for the scalar product of two real vectors $\mathbf{x}, \mathbf{y} \in \mathcal{R}^{2N}$ and

$$< \mathbf{w}, \mathbf{z} > = \sum_{n=1}^{N} w_n^* z_n \tag{2}$$

for the scalar product of two complex vectors $\mathbf{w}, \mathbf{z} \in \mathcal{C}^N$. If $\mathbf{w} \leftrightarrow \mathbf{x}$ and $\mathbf{z} \leftrightarrow \mathbf{y}$, there is a relation between the corresponding scalar products given by

$$\mathbf{x} \cdot \mathbf{y} = \Re\{< \mathbf{w}, \mathbf{z} >\}. \tag{3}$$

Let $\mathbf{s} = (s_1, ..., s_K)^{\mathrm{T}} \in \mathcal{C}^K$ be a vector of complex symbols that carry the information to be transmitted and define $\mathbf{x}$ by $\mathbf{s} \leftrightarrow \mathbf{x}$. A space-time code from a *generalized orthogonal design* [3] of rate $R = K/N$ for M transmit antennas is a linear mapping $\mathbf{x} \mapsto \mathbf{S}$ (or, equivalently, $(\mathbf{s}, \mathbf{s}^*) \mapsto \mathbf{S}$) from $\mathcal{R}^{2K}$ into the space of (complex) $N \times M$ matrices with the property

$$\mathbf{S}^{\dagger}\mathbf{S} = \|\mathbf{s}\|^2 \mathbf{I}_N \tag{4}$$

for every symbol vector $\mathbf{s} \leftrightarrow \mathbf{x}$. Here $\mathbf{S}^{\dagger}\mathbf{S}$ denotes the Hermitian conjugate of $\mathbf{S}$. Eq. (4) means that all the columns of the *code matrix* $\mathbf{S}$ are orthogonal. Note that for (proper square matrix) orthogonal designs, the matrix $\mathbf{S}$ is square ($K = N$), while for all other cases of generalized orthogonal designs, $K < N$ holds.

Following [6], we may express $\mathbf{S}$ as a linear combination of *base matrices* $\{\beta_k\}_{k=1}^{2K}$:

$$\mathbf{S} = \sum_{k=1}^{2K} x_k \beta_k \tag{5}$$

Then, eq. (4) holds if and only if the base matrices have the property

$$\beta_k^\dagger \beta_l + \beta_l^\dagger \beta_k = 2\delta_{kl}\mathbf{I}_N, \tag{6}$$

where $\mathbf{I}_N$ is the $N \times N$ identity matrix.

III. Complex and Real Transmission Setup

In the transmission setup, the mth column of the code matrix is transmitted at antenna number m and is attenuated by a complex fading coefficient c_m. The fading is assumed to be constant during the transmission of the N symbols in each column. The vector of received symbols can then be written as

$$\mathbf{r} = \mathbf{S}\mathbf{c} + \mathbf{n}_c. \tag{7}$$

Here $\mathbf{c} = (c_1, ..., c_M)^{\mathrm{T}}$ is the vector of complex multiplicative fading coefficients, and $\mathbf{n}_c$ is complex AWGN with variance $\sigma^2 = N_0/2$ in each real dimension. We use eq. (5) to write

$$\mathbf{S}\mathbf{c} = \sum_{k=1}^{2K} x_k \mathbf{h}_k \tag{8}$$

as a linear combination of channel-dependent transmit base vectors given by $\mathbf{h_k} = \beta_\mathbf{k}\mathbf{c}$. Using eq. (6), we can show that

$$\Re\{< \mathbf{h}_k, \mathbf{h}_l >\} = ||\mathbf{c}||^2 \delta_{\mathbf{kl}}. \tag{9}$$

We, thus, see that even though the complex base vectors $\{\mathbf{h}_k\}_{k=1}^{2K}$ are not orthogonal, they correspond to a real orthogonal base. Let $c = ||\mathbf{c}||$ and define the corresponding normalized real base $\{\mathbf{g}_k\}_{k=1}^{2K}$ by $c^{-1}\mathbf{h}_k \leftrightarrow \mathbf{g}_k$. This base is *orthonormal*, i.e.,

$$\mathbf{g}_k \cdot \mathbf{g}_l = \delta_{kl}. \tag{10}$$

Defining the real receive vector $\mathbf{y} \in \mathcal{R}^{2N}$ by $\mathbf{r} \leftrightarrow \mathbf{y}$ and the real AWGN vector $\mathbf{n} \in \mathcal{R}^{2N}$ by $\mathbf{n}_c \leftrightarrow \mathbf{n} \in \mathcal{R}$, the transmission can be re-written in an equivalent real model by

$$\mathbf{y} = c \sum_{k=1}^{2K} x_k \mathbf{g}_k + \mathbf{n}. \tag{11}$$

With the definition $\mathbf{G} = [\mathbf{g}_1, ..., \mathbf{g}_{2K}]$, this can be written in the more compact matrix form as

$$\mathbf{y} = c\,\mathbf{G}\mathbf{x} + \mathbf{n}. \tag{12}$$

Since the real transmit base vectors are orthonormal, the matrix $\mathbf{G}$ has the property

$$\mathbf{G}^{\mathrm{T}}\mathbf{G} = \mathbf{I}_{2K}. \tag{13}$$

This means that $\mathbf{G}$ leaves distances and angles invariant, even though, for $K < N$, it is not orthogonal. Neverthess, as we will see below, it can be interpreted as a kind of rotation of a lower- dimensional plane into a higher-dimensional space.

IV. The Optimum Receiver

We assume perfect channel state information at the receiver. As a result of the well- established classical detection theory (see, e.g., [9]), we recall that for a transmission model given by eq. (11), the scalar products $v_k = \mathbf{g}_k \cdot \mathbf{y}$ of the receive vector with the base vectors form a set of sufficient statistics, i.e., they comprise all the information that is relevant for the optimum receiver. We write these values as a vector $\mathbf{v} = (v_1, ..., v_{2K})^{\mathrm{T}}$ that will be calculated from the receive vector $\mathbf{y}$ as $\mathbf{v} = \mathbf{G}^{\mathrm{T}}\mathbf{y}$. Inserting this into eq. (12) and using eq. (13), we obtain

$$\mathbf{v} = c\,\mathbf{x} + \mathbf{m}, \tag{14}$$

where, as a consequence of eq. (13), the transformed vector $\mathbf{m} = \mathbf{G}^{\mathrm{T}}\mathbf{n}$ is $2K$-dimensional white Gaussian noise with variance $\sigma^2 = N_0/2$ in each dimension. We thus have a tranmission model with a signal only attenuated by a composed real fading amplitude c and additive white Gaussian noise. Since $c^2 = c_1^2 + ... + c_M^2$, eq. (14) is equivalent to the conventional M-fold receive antenna diversity with the maximum ratio combining (MRC) receiver, and it can be analysed with the same methods, see e.g. [10].

We note that a least squares condition on eq. (12) leads to the same result for the receiver. Because of eq. (13), the vector $\mathbf{x}$ that minimizes the squared Euclidean distance $||\mathbf{y} - c\,\mathbf{G}\mathbf{x}||^2$ is the same that maximizes the expression $2\mathbf{G}^{\mathrm{T}}\mathbf{y} \cdot \mathbf{x} - c^2||\mathbf{x}||^2$. With $\mathbf{v} = \mathbf{G}^{\mathrm{T}}\mathbf{y}$, this is the same vector that minimizes $||\mathbf{v} - c\,\mathbf{x}||^2$, which is the least squares conditon on eq. (14). Therefore, the least squares receiver for eq. (12) first calculates $\mathbf{v} = \mathbf{G}^{\mathrm{T}}\mathbf{y}$ and then looks for the least mean squares solution of the equivalent AWGN channel given by eq. (14).

V. Geometrical Interpretation

The following geometrical arguments are depicted in Fig. 1 for the value $c = 1$ of the composed real fading amplitude. The base $\{\mathbf{g}_k\}_{k=1}^{2K}$ spans a $2K$-dimensional subspace of $\mathcal{R}^{2N}$ that we denote by $\mathcal{G}$. We visualize this subspace as a $2K$-dimensional *plane* that is embedded in the $2N$-dimensional *space* $\mathcal{R}^{2N}$. Because $\mathcal{G}$ is the image of the norm-preserving linear map $\mathbf{G}$, i.e., $\mathcal{G} = \mathbf{G}\mathcal{R}^{2K}$, we visualize $\mathcal{G}$ as a rotation of the plane $\mathcal{R}^{2K}$ into the space $\mathcal{R}^{2N}$. The matrix $\mathbf{G}^{\mathrm{T}}$ acts on $\mathcal{G}$ as a back-rotation of every vector $\mathbf{G}\mathbf{x}$ to its original vector $\mathbf{x}$, see Fig. 1. The real receive vector $\mathbf{y}$ in eq. (12), however, as the sum of a vector in $\mathcal{G}$ and a $2N$-dimensional noise vector $\mathbf{n}$, is typically not in $\mathcal{G}$. The question is how $\mathbf{G}^{\mathrm{T}}$ acts on $\mathbf{y}$. From the dectection theory we know that, in a Gaussian noise channel, all the components of the receive signal that are orthogonal to the vector space of the transmit signals are irrelevant for the decision (see e.g [9], p. 84f). If we decompose $\mathbf{y} = \mathbf{y}_{||} + \mathbf{y}_\perp$ into the uniquely defined components parallel and perpendicular to $\mathcal{G}$, resp., then the decisions do not depend on $\mathbf{y}_\perp$. In fact, $\mathbf{G}^{\mathrm{T}}\mathbf{y}_\perp = 0$ holds because for any real matrix $\mathbf{A}$, the identity $image(\mathbf{A})^\perp = kernel(\mathbf{A}^{\mathrm{T}})$ holds. Therefore, $\mathbf{G}^{\mathrm{T}}\mathbf{y} = \mathbf{G}^{\mathrm{T}}\mathbf{y}_{||}$, and the action of $\mathbf{G}^{\mathrm{T}}$ on $\mathbf{y}$ can be visualized as a two-step operation: First, as a projection of $\mathbf{y}$ onto $\mathcal{G}$, resulting in $\mathbf{y}_{||}$. Then $\mathbf{y}_{||} \in \mathcal{G}$ will be back-rotated to $\mathcal{R}^{2K}$ where the receiver takes its decisions. This illustrates why the transmission setup is equivalent to the setup given by eq. (14), where the transmission stays inside the plane $\mathcal{R}^{2K}$: It is because the rotation and back-rotation cancel out each other, and the perpendicular components of the noise are irrelevant for the decision.

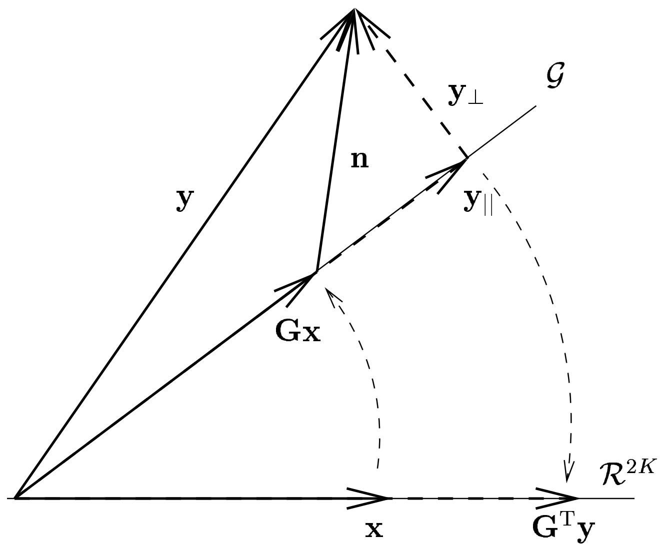

Fig. 1. Geometrical model for the transmission channel as rotation and back-rotation.

Fig. 1 suggests the comparison with a BPSK transmission over a channel with a phase rotation by an angle φ and an amplitude attenuation by a factor c. In fact, this setup turns out to be a special case of the same formalism: In that case, the vector $\mathbf{x}$ reduces to a one-dimensional BPSK symbol $x \in \{\pm 1\}$, and the phase rotation by the angle φ is described by the 2×1 matrix $\mathbf{G} = (\cos\varphi, \sin\varphi)^T$ that maps the one-dimensional BPSK symbol into $\mathcal{G}$, a one-dimensional subspace of the two-dimensional quadrature space. The receiver takes its decision on the backrotated receive symbol, thereby ignoring the component perpendicular to $\mathcal{G}$. We may therefore interpret the (generalized) orthogonal space-time code transmission as a generalized phase-rotation into a higher- dimensional signal space, together with the attenuation by a composed fading amplitude.

VI. Concluding Remarks

We have presented a simple geometrical interpretation of orthogonal space-time code transmission and we have shown how the optimum receiver can be derived by an intuitively obvious geometrical argument. In our treatment, we have consciously not used the established complex formalism. Only in a real vector space we are able to give such a simple geometrical interpretation of the signals [1]. This is because the signal vector $\mathbf{Cs}$ transmitted from the antennas is not a complex linear function of the K complex modulation symbols $\{s_k\}_{k=1}^{K}$. It is linear in the $2K$ complex symbols $\{s_k, s_k^*\}_{k=1}^{K}$, or equivalently, linear in the $2K$ real symbols $\{x_k\}_{k=1}^{2K}$. We thus have to deal with an essentially real linear transmission setup that has its natural interpretation in a real vector space of signals.

We will now show how the complex-notation receiver derived by other authors can be obtained from our real-notation receiver. We define a complex vector $\mathbf{u} \leftrightarrow \mathbf{v}$ by writing $\mathbf{u} = \mathbf{v}' + j\mathbf{v}''$ with components $u_k = v_k' + jv_k''$ for $k = 1, ..., K$. We then write the scalar products $v_k = \mathbf{g}_k \cdot \mathbf{y}$ as the real parts of complex scalar products, using the correspondences $c^{-1}\mathbf{h}_k \leftrightarrow \mathbf{g}_k$ and $\mathbf{r} \mapsto \mathbf{y}$. With the definitions $\beta_k' = \beta_k$ and $\beta_k'' = \beta_{k+K}$ for $k = 1, ..., K$ we eventually obtain

$$c\,u_k = \Re\{< \beta_k'\mathbf{c}, \mathbf{r} >\} + j\Re\{< \beta_k''\mathbf{c}, \mathbf{r} >\}. \quad (15)$$

This is just the same as eq. (12) in [5]. The expressions given by [6] and [7] can also be derived from this equation. The receiver in this complex notation does not have such a simple geometrical interpretation. Furthermore, because signal processing in a practical receiver will be done with the real and imaginary parts, it seems to be quite natural here use the real notation.

[1]Except for the Alamouti scheme for two transmit antennas, where a complex treatment with a geometrical interpretation is possible [8].

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