Spatial Diversity and Eigenspectra

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Overview and Questions

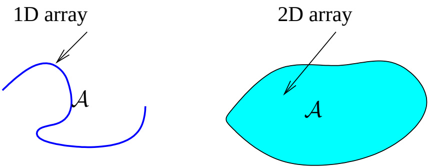
- How much diversity can be achieved by a given antenna array geometry (aperture) ?
- Consider *dense antenna arrays* (apertures) with spatial correlations
- Calculation of the eigenvalue spectrum of the autocorrelation kernel
- This diversity spectrum depends ond the aperture and the power azimuth spectrum (PAS)
- The diversity measure (Ivrlač, Nossek) summarises its most important properties

Goal: Quantitative statements about the diversity of a given combination of aperture geometry and PAS

The Concept of a Dense Array (Aperture) $\mathcal{A}$

Visual Concept: Continuous limit of an array densely covered by infinitely many small sensors

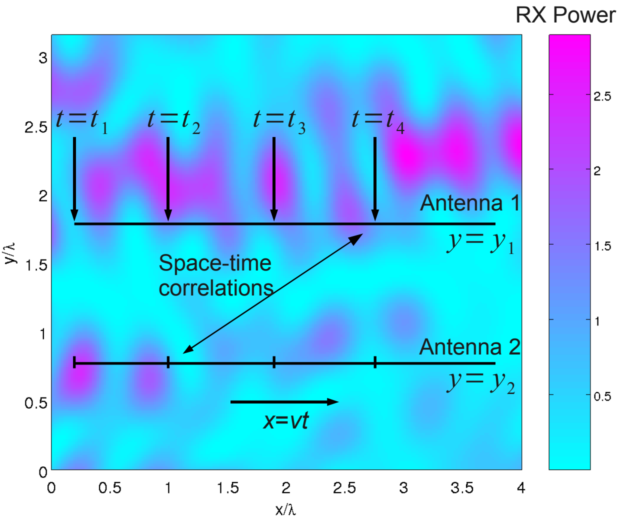
Mathematical Concept: Continuous stochastic process (fading) on a geometrical object $\mathcal{A}$ (1D or 2D) in the plane



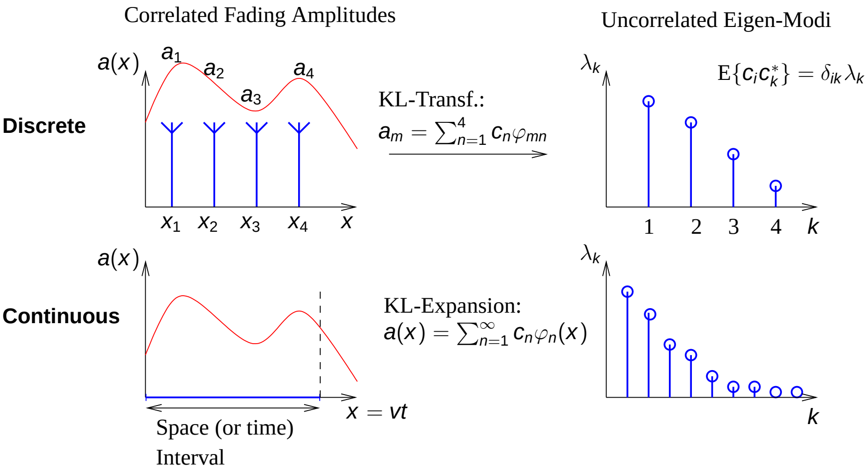
$\rightsquigarrow$ Calculate the diversity of the fading process defined on any given aperture !

Time Variance of Fading = Scaled Space Variance

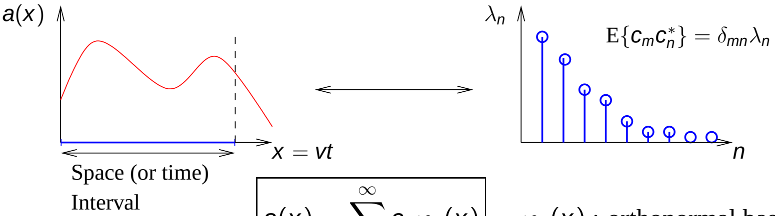
Aperture $\mathcal{A}$ for 2 antennas on a moving vehicle: 2 parallel line segments (2 ULAs)



Karhunen-Loève (KL) Expansion



$\{\lambda_n\}$: Eigenvalue spectrum of the autocorrelation matrix (discrete case) or integral kernel (continuous case) of the fading



$$a(x) = \sum_{n=1}^{\infty} c_n \varphi_n(x)$$

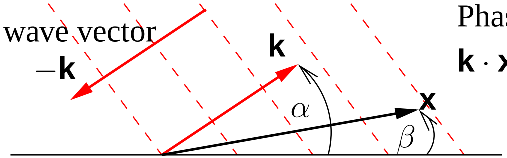
$\varphi_n(x)$: orthonormal basis

Interpretation:

- Equivalent uncorrelated diversity branches, each corresponding to one term in the sum
- $\lambda_n = \mathrm{E}\left\{|c_n|^2\right\} \leftrightsquigarrow$ power gain of branch number n
- We shall see: For a finite array $\mathcal{A}$, only a finite number of terms in the sum are relevant ($\leftrightsquigarrow$ eigenvalues λ_n of significant size)
- $\rightsquigarrow$ Finite degree of diversity for a finite array $\mathcal{A}$

Spatial 2D Fading Process

Plane wave (wavelength=1) impinging from direction α :



Phase at point $\mathbf{x}$:

$$\mathbf{k} \cdot \mathbf{x} = 2\pi r \cos(\alpha - \beta)$$

$$\mathbf{k} = 2\pi \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} \qquad \mathbf{x} = r \begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix}$$

Spatial fading process = superposition of many plane waves :

$$a(\mathbf{x}) = \int_{-\pi}^{\pi} \hat{a}(\alpha)\, e^{j2\pi r \cos(\alpha-\beta)} d\alpha$$

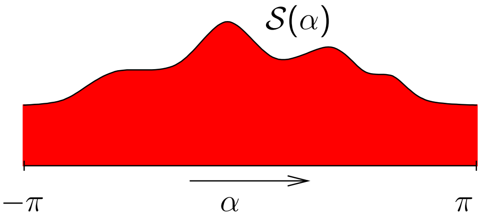
$\hat{a}(\alpha)$: scattering amplitude of the wave from direction α

Spatial Fading Statistics

Assumption: *Uncorrelated Scattering* (US) Amplitudes:

$$\mathrm{E}\left\{\hat{a}\left(\alpha\right)\hat{a}^{*}\left(\alpha'\right)\right\} = \mathcal{S}\left(\alpha\right)\delta\left(\alpha-\alpha'\right)$$

$\mathcal{S}\left(\alpha\right) =$ Power Azimuth Spektrum (PAS): Known from the model



US $\Leftrightarrow$ Translational invariance of the autocorrelation:

$$\mathrm{E}\left\{a\left(\mathbf{x}\right)a^{*}\left(\mathbf{x}'\right)\right\} = \rho\left(\mathbf{x}-\mathbf{x}'\right)$$

$$\rho\left(\mathbf{x}\right) = \int_{-\pi}^{\pi} e^{j2\pi r\cos(\alpha-\beta)}\mathcal{S}\left(\alpha\right)\mathrm{d}\alpha$$

Fourier Expansions and the PAS Matrix $\widetilde{\mathbf{R}}$

Expansions for the scattering amplitude and for the PAS:

$$\hat{a}(\alpha) = \sum_{n=-\infty}^{\infty} \hat{a}_n e^{j\alpha n} \quad \text{and} \quad \tilde{a}_n = 2\pi\hat{a} = \int_{-\pi}^{\pi} \mathrm{e}^{-j\alpha n} \hat{a}(\alpha)\, \mathrm{d}\alpha$$

$$\mathcal{S}(\alpha) = \sum_{n=-\infty}^{\infty} \hat{s}_n e^{j\alpha n} \quad \text{and} \quad \tilde{s}_n = 2\pi\hat{s} = \int_{-\pi}^{\pi} \mathrm{e}^{-j\alpha n} \mathcal{S}(\alpha)\, \mathrm{d}\alpha$$

Uncorrelated scattering $\Rightarrow$ These coefficients are related by

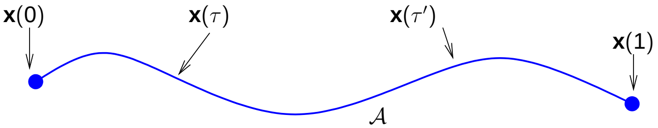
$$\tilde{R}_{mn} \triangleq \mathrm{E}\left\{\tilde{a}_m \tilde{a}_n^*\right\} = \tilde{s}_{m-n}$$

$\rightsquigarrow$ *PAS (correlation) matrix* $\widetilde{\mathbf{R}}$ (Toeplitz) with elements $\tilde{R}_{mn}$ gives an equivalent descriptions of the scattering environment:

$$\mathcal{S}(\alpha) \leftrightsquigarrow \widetilde{\mathbf{R}}$$

The KLE on a 1D Array (Curve in the Plane)

$\mathbf{x}(\tau)$: Parametrisation of the curve $\mathcal{A}$. Parameter $\tau \in [0, 1]$



$a(\tau) \triangleq a(\mathbf{x}(\tau))$: Fading process on the curve

KL Expansion of the fading process on the curve:

$$a(\tau) = \sum_{n=1}^{\infty} c_n \varphi_n(\tau)$$

$$c_n = \int_0^1 \varphi_n^*(\tau)\, a(\tau)\, \mathrm{d}\tau\,, \quad \mathrm{E}\{c_m c_n^*\} = \lambda_n \delta_{mn} \text{ with}$$

eigenvalues λ_n and eigenfunctions $\varphi_n(\tau)$ of the integral kernel

$$R(\tau, \tau') = \mathrm{E}\left\{a(\tau)\, a^*(\tau')\right\} = \rho\left(\mathbf{x}(\tau) - \mathbf{x}(\tau')\right)$$

How to Solve the Eigenvalue Problem: Transformation

Non-orthogonal expansion of the spatial fading process:

$$a(\mathbf{x}) = \sum_{n=-\infty}^{\infty} \tilde{a}_n v_n(\mathbf{x}) \quad , \quad v_n(\mathbf{x}) = \mathrm{e}^{j\beta n} j^n \mathrm{J}_n(2\pi r)$$

Define Gram matrix **G** with elements

$$G_{mn} = \int_0^1 v_m^*(\tau)\, v_n(\tau)\, \mathrm{d}\tau \,, \quad \left(v_n(\tau) \triangleq v_n(\mathbf{x}(\tau)) \right)$$

Theorem

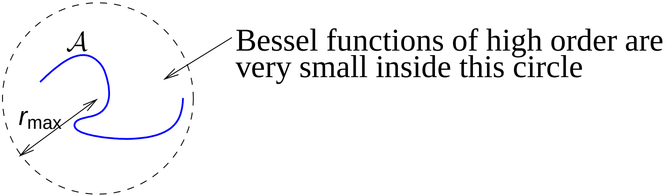
The kernel $R(\tau, \tau')$ *has the same eigenvalues as the infinite-dimensional matrix*

$$\tilde{\mathbf{R}}\mathbf{G}$$

Scattering Environment $\mathcal{S}(\alpha)$ → $\tilde{\mathbf{R}}\mathbf{G}$ ← Array Geometry $\mathcal{A}$

How to Solve the Eigenvalue Problem: Truncation

Assume that $\mathcal{A}$ is inside a finite circle



$$\Rightarrow \; |G_{mn}| \leq \int_0^1 |\mathrm{J}_m(2\pi |\mathbf{x}(\tau)|)| \, |\mathrm{J}_n(2\pi |\mathbf{x}(\tau)|)| \, \mathrm{d}\tau \to 0$$

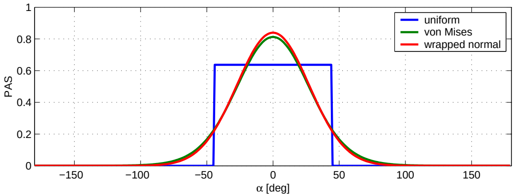
(exponentially fast) for $|m|, |n| \to \infty$

Theorem

When the infinite matrix $\tilde{\mathbf{R}}\mathbf{G}$ *is replaced by a truncated finite one, the effect on the eigenvalues is exponentially small.*

$\rightsquigarrow$ Solve the EVP numerically for the truncated matrix

PAS Models $\mathcal{S}(\alpha)$ (and $\mathcal{S}(\alpha - \alpha_0)$ for AoA α_0)



1. Uniform with width Δ $\longleftarrow$ concentrate on this example

$$\mathcal{S}(\alpha) = \mathrm{rect}\left(\frac{\alpha}{\Delta}\right) \Rightarrow \tilde{s}_n = \mathrm{sinc}\left(n\frac{\Delta}{2\pi}\right)$$

2. von-Mises with dispersion κ^{-1}

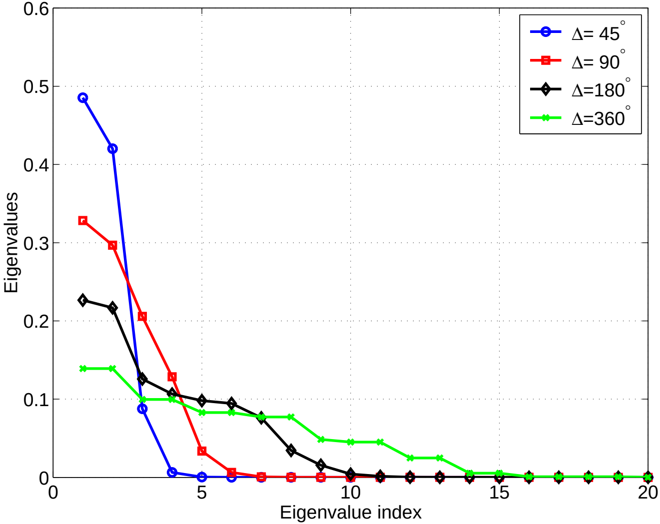
$$\mathcal{S}(\alpha) = \frac{\exp(\kappa \cos \alpha)}{2\pi \mathrm{I}_0(\kappa)} \Rightarrow \tilde{s}_n = \frac{\mathrm{I}_n(\kappa)}{\mathrm{I}_0(\kappa)}$$

3. Wrapped Normal with dispersion σ^2

$$\mathcal{S}(\alpha) = \frac{1}{\sqrt{2\pi}\,\sigma} \sum_{k=-\infty}^{\infty} \mathrm{e}^{-\frac{(\alpha - 2\pi k)^2}{2\sigma^2}} \Rightarrow \tilde{s}_n = \mathrm{e}^{-\frac{1}{2}\sigma^2 n^2}$$

Calculated Diversity Spectra

Uniform circular array (UCA) of radius $r = 1$, uniform PAS



Diversity Measure

Ivrlač, Nossek 2003; Muharemovic et al. 2004/2008

Attempt to summarise the main properties of $\{\lambda_n\}$ by a single number ω

For L diversity branches:

$$\omega \triangleq \frac{\left(\sum_{i=1}^{L} \lambda_i\right)^2}{\sum_{i=1}^{L} \lambda_i^2}$$

$\omega = L$ for uncorrelated branches and $\omega = 1$ for 100% correlation

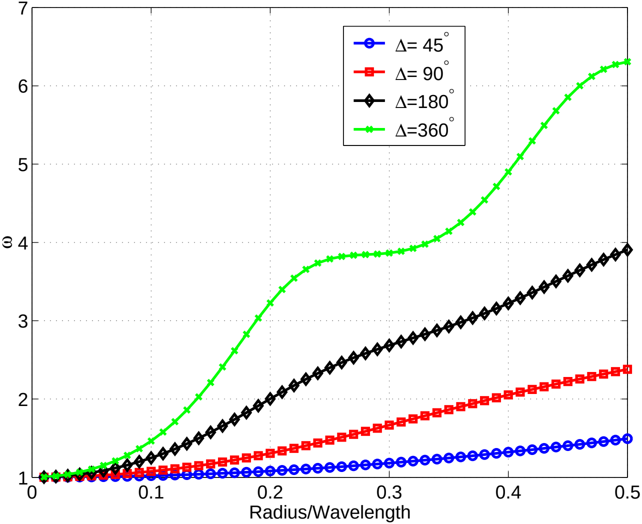
Lozano et al. 2003; Muharemovic 2004/2008: ω determines the capacity slope of a MIMO system in the low power regime

We extend the definition to a continuous aperture:

$$\omega \triangleq \frac{(\sum_{i=1}^{\infty} \lambda_i)^2}{\sum_{i=1}^{\infty} \lambda_i^2}$$

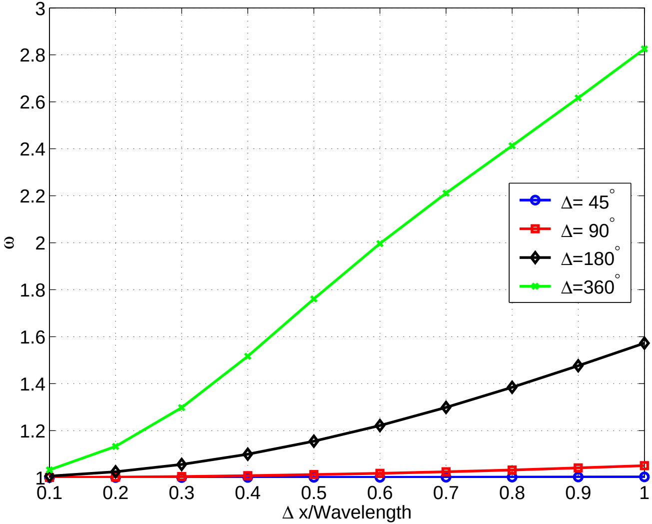
Diversity Measure as a Function of the Array Size

Uniform circular array (UCA), uniform PAS



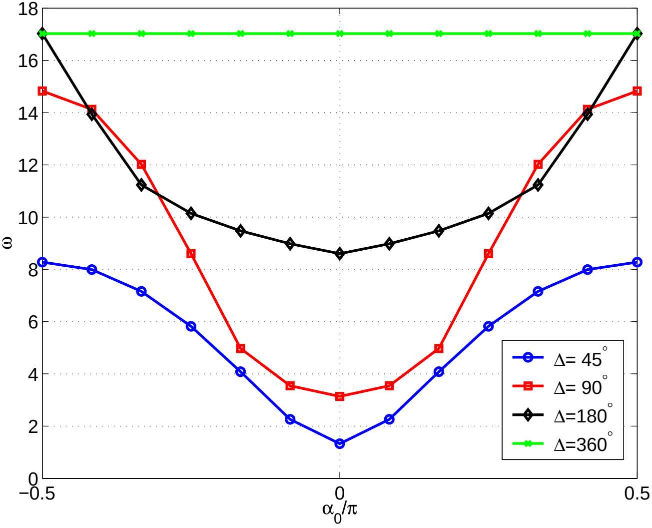
Diversity Measure as a Function of the Array Size

Uniform linear array (ULA), uniform PAS, AoA $\alpha_0 = 0$ (signal || array)

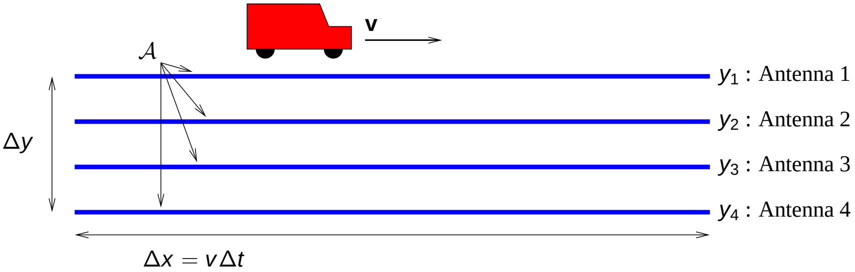


Diversity Measure as a Function of the AoA α_0

Large uniform linear array (ULA) of size $\Delta x = 10$, uniform PAS

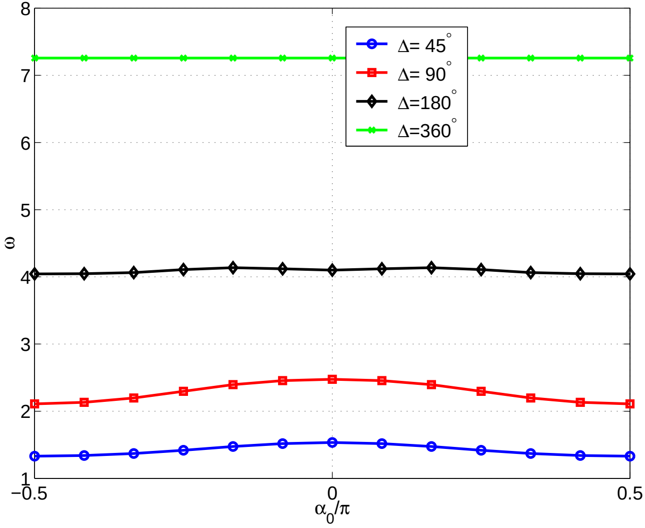


4 Parallel Dense ULAs
$\Leftrightarrow$ 4 Antennas (Traverse) on a Moving Vehicle



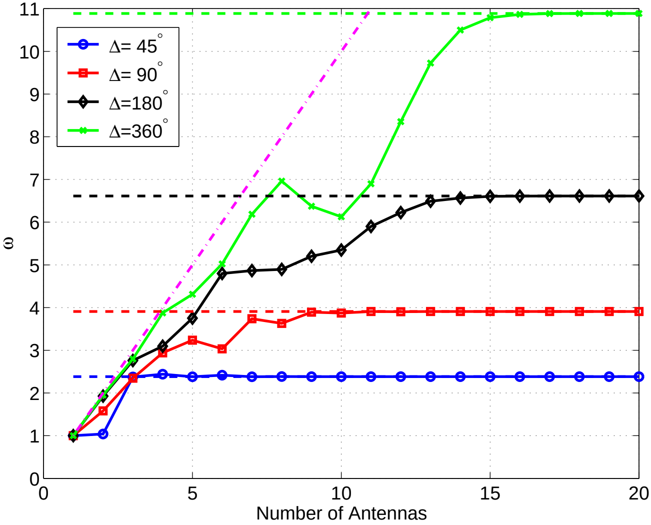
Diversity Measure as a Function of the AoA α_0

4 parallel ULAs of aperture size $\Delta x = 1$, $\Delta y = 1$, uniform PAS



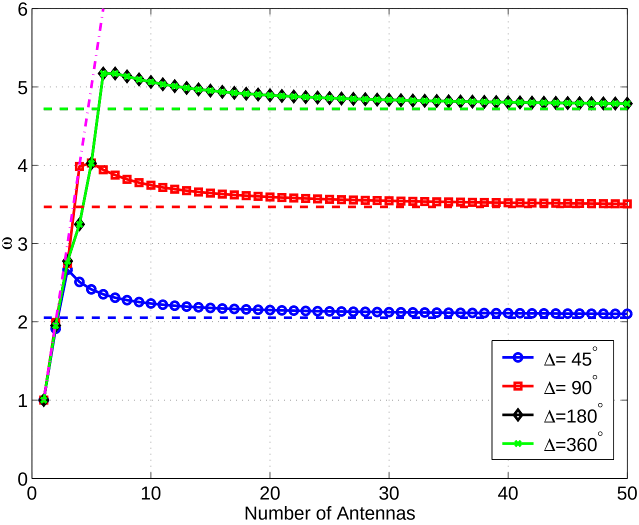
Diversity Measures: Discrete vs. Dense (- -) UCAs

$r = 1$, uniform PAS; approximation: $\rho(\mathbf{x}) \approx \sum_{n=-N}^{N} \tilde{s}_n e^{j\beta n} j^n \mathrm{J}_n\left(2\pi \frac{r}{\lambda}\right)$



Diversity Measures: Discrete vs. Dense (- -) ULAs

$\Delta x = 2$, uniform PAS, $\alpha_0 = 90°$; approximation: $\rho(\mathbf{x}) \approx \sum_{n=-N}^{N} \tilde{s}_n e^{j\beta n} j^n \mathrm{J}_n\left(2\pi \frac{r}{\lambda}\right)$



Conclusion

- A method has been developed to calculate diversity spectra and diversity measures for general dense arrays
- Diversity measures determine the MIMO capacity in the low power regime
- The results are useful to optimise the antenna arrangement on a vehicle $\rightsquigarrow$
- Traversely mounted antenna arrays on a vehicle are better than parallel structures
- Time interleaving and spatial diversity are entangled $\Rightarrow$ antenna separation and time interleaving depth should be jointly designed