The Influence of Spatial Correlations on a Coded OFDM System with Antenna Diversity

Henrik Schulze
University of Applied Sciences South Westphalia
Lindenstr. 53, D-59872 Meschede, Germany
Email: schulze@fh-swf.de

***Abstract*— For a two-antenna diversity system with BPSK or QPSK modulation and channel coding, we present a simple rule-of-thumb formula for the SNR loss due to spatial correlations. This formula assumes perfect interleaving, but it does not include any parameters of the special code.**

I. Introduction

Antenna diversity is an useful and popular technique to improve the performance of an OFDM system in a mobile radio channel. In practice, however, a large spatial separation of the antennas is difficult to realize. Spatial correlations between the channel coefficients at the respective antenna locations may then lead to some performance degradations of the diversity system. Since an analytical investigation is often too complicated, computer simulations of the whole system are the proper way to quantify these degradations. To perform such simulations, much time has to be spent on the development of the software. For this reason, it is quite helpful to have a simple rule of thumb for the SNR (signal-to-noise ratio) loss due to the finite antenna separation.

In this paper, we show how a simple estimate of the SNR loss can be obtained from the error event probabilities of a convolutionally coded BPSK or QPSK system in a Rayleigh fading channel. We consider only two receive (or two transmit) antennas, and we assume ideal interleaving in time and frequency. For this case, by using some approximations, we are able to find an expression that provides us directly with the SNR loss depending on the spatial correlation coefficient. It is interesting to note that – for a given correlation coefficient – this loss depends only on the value of the SNR, but not on the special parameters of the coding.

The remainder of this paper is organized as follows: In Section II, we describe the transmission setup and the theoretical error event probabilities. In Section III, we use these formulas to obtain a simple expression for the SNR loss due to the spatial correlations between two antenna. In Section IV, we evaluate these expression for a concrete example. Finally, in Section V, we draw some conclusions.

II. Transmission Setup and Error Event Probabilities

We consider a convolutionally coded OFDM transmission scheme of code rate R_c with BPSK or QPSK modulation in a time-variant Rayleigh fading channel. The symbols take values $s_i \in \{\pm\Delta\}$ for BPSK and $s_i \in \{\pm\Delta \pm j\Delta\}$ for QPSK ($\Delta > 0$). The symbol energy is given by

$$E_S = \Delta^2 \log_2 M \,, \tag{1}$$

where $M = 2$ for BPSK and $M = 4$ for QPSK.

We consider a number of L_r receive antennas and apply maximum ratio combining (MRC) [1]. Alternatively (or additionally) we may also consider Alamouti's two-antenna transmit diversity scheme [2]. Let the number of transmit antennas be denoted by L_t, i.e. $L_t = 2$ if the Alamouti scheme is applied and $L_t = 1$ if not. We further write

$$L = L_r L_t \tag{2}$$

for the maximal diversity degree that can be obtained by the antennas.

For a fixed time slot and a fixed OFDM subcarrier index, there are L complex fading amplitudes $a_i \quad (i = 1, ..., L)$ corresponding to the L paths between the L_t transmit and L_r receive antennas. Applying the usual WSSUS (wide-sense-stationary uncorrelated scattering) assumption, the second-order statistics of the fading amplitudes is time and frequency independent. The spatial correlation coefficient between the antennas is given by the expectation value

$$\rho_{ik} = \mathrm{E}\{a_i a_k^*\} \,. \tag{3}$$

For each transmission path number $i \quad (i = 1, ..., L)$, the parameter ρ_{ii} can be interpreted as the *channel gain factor*. We assume that every transmission path has the same path loss. We may then normalize all the channel gain factors to

$$\rho_{ii} = 1\,. \tag{4}$$

The (average) energy per bit at the receiver, E_b, is related to the symbol energy, E_S, by

$$L\, E_S = R_c \log_2 (M)\; E_b \tag{5}$$

For simplicity, the loss due to the OFDM guard interval has been discarded in this equation.

Let N_0 be the one-sided noise density. The signal-to-noise ratio $SNR = L_t E_S / N_0$ at each receive antenna is given by

$$SNR = L_t \log_2 (M) \cdot \gamma\,, \tag{6}$$

where we have used the abbreviation

$$\gamma = \frac{\Delta^2}{N_0} = \frac{1}{\log_2 M} \frac{E_S}{N_0}\,. \tag{7}$$

Let us now consider maximum likelihood decoding of a given code. We assume ideal interleaving in time and frequency. This means that the fading amplitudes corresponding to different symbols of the same codeword are uncorrelated. By contrast, we assume correlations between the fading amplitudes that correspond to the same symbol received (or transmitted) at different antennas. For the techniques to analyse correlated fading amplitudes, we refer to Subsection 4.4.3 in [3]. In [4], we have applied these methods for the reverse case with uncorrelated fading at the antennas but correlated fading for the codeword symbols. Equation (40) in that papers also holds for the case considers here, but Equation (41) for the probability P_d of an error event of Hamming distance d has to be replaced by[1]

$$P_d = \frac{1}{\pi} \int_0^{\pi/2} \prod_{i=1}^{L} \left(\frac{1}{1 + \frac{\lambda_i \gamma}{\sin^2 \theta}} \right)^d \mathrm{d}\theta\,. \tag{8}$$

In that equation, the λ_i are the eigenvalues of the spatial autocorrelation matrix $\mathbf{R}$ with elements ρ_{ik}. An approximate expression for the bit error probability P_b can be obtained from the error event probability in Equation (8) by using the tight union bound

$$P_b \leq \sum_{d=d_{dree}}^{\infty} c_d P_d\,. \tag{9}$$

[1]In fact, the only modification is that the parameters L and d exchange their roles.

The coefficients c_d are the error coefficients of the code. For the code used in this paper, they are tabulated in [5].

If the autocorrelation matrix $\mathbf{R}$ is known, approximate bit error curves can be obtained from Equations (8) and (9) for any number of receive antennas and for one or two transmit antennas (i. e. with or without the Alamouti setup). Typically, for higher values of L, the eigenvalues of the correlation matrix have to be calculated by numerical methods.

III. A Simple Expression for the SNR Loss

For the special case $L = 2$ (i. e. only two-fold antenna diversity either at the receiver or at the transmitter), we will now analyse Equation (8) further to obtain a simple approximate estimation of the SNR loss due to spatial antenna correlations. For $L = 2$, Equation (8) reduces to

$$P_d = \frac{1}{\pi} \int_0^{\pi/2} \left(\frac{1}{1 + \frac{\lambda_1 \gamma}{\sin^2 \theta}} \cdot \frac{1}{1 + \frac{\lambda_2 \gamma}{\sin^2 \theta}} \right)^d \mathrm{d}\theta. \tag{10}$$

The eigenvalues λ_1 and λ_2 of the 2×2 autocorrelation matrix

$$\mathbf{R} = \begin{pmatrix} 1 & \rho_{12} \\ \rho_{12}^* & 1 \end{pmatrix} \tag{11}$$

can be directly calculated as

$$\lambda_1 = 1 + |\rho_{12}| \tag{12}$$

$$\lambda_2 = 1 - |\rho_{12}|\,. \tag{13}$$

We insert these value into Equation (10) and obtain the following expression:

$$P_d = \frac{1}{\pi} \int_0^{\pi/2} \left(\frac{1}{\left(1 + \frac{\gamma}{\sin^2 \theta}\right)^2 - |\rho_{12}|^2 \left(\frac{\gamma}{\sin^2 \theta}\right)^2} \right)^d \mathrm{d}\theta \tag{14}$$

For further analysis, we upper-bound the integral in Equation (14) by

$$P_d \leq \frac{1}{2 \left[f_\rho (\gamma) \right]^d} \tag{15}$$

with

$$f_\rho (\gamma) = (1 + \gamma)^2 - \rho^2 \gamma^2\,. \tag{16}$$

Due to Equation (6), the SNR is proportional to γ. We may thus use inequality (15) to get an estimate of the SNR loss due to the non-zero correlation coefficient ρ. We assume that an SNR loss in the expression on the right-hand side of that inequality results in approximately the same loss for the left-hand side. This has the following consequence: If for $\rho = 0$ some value $\gamma = \gamma_0$ is needed to achieve a certain value of P_d , and for and $\rho > 0$ another

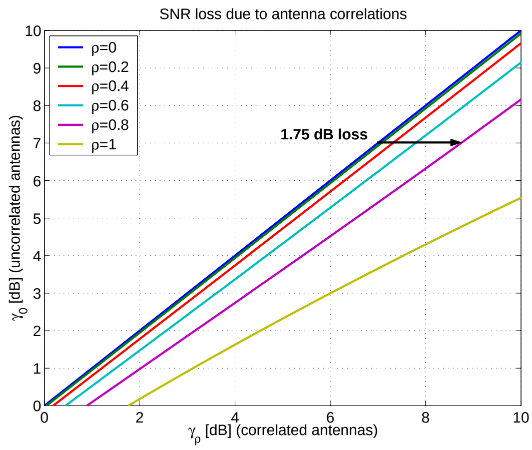

Fig. 1. SNR loss due to antenna correlations.

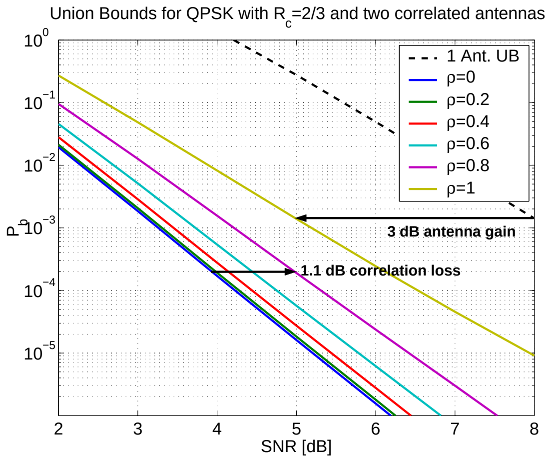

Fig. 2. Union bounds (UBs) for the bit error probability for 2 receive antennas.

(higher) value $\gamma = \gamma_\rho$ is needed to achieve the same value of P_d, the corresponding values of the function (16) are related by

$$f_0(\gamma_0) = f_\rho(\gamma_\rho) \,. \tag{17}$$

We insert Equation (16), solve for γ_0, and obtain

$$\gamma_0 = \sqrt{(1+\gamma_\rho)^2 - \rho^2\gamma_\rho^2} - 1 \,. \tag{18}$$

For a given set of parameters ρ, one can now plot curves for γ_0 as a function of γ_ρ and read the SNR loss as the horizontal distance to the curve corresponding to $\rho = 0$. Figure 1 shows these curves for $\rho = 0, 0.2, 0.4, 0.6, 0.8, 1$. At $\gamma_0 = 7$ dB, for example, there is a horizontal difference of approx. 1.75 dB between the blue curve corresponding to $\rho = 0$ and the magenta curve corresponding to $\rho = 0.8$. This 1.75 dB difference is the SNR loss due to spatial correlation.

The SNR is related to γ by Equation (6). For receive antenna diversity ($L_r = 2$ and $L_t = 1$), $SNR = \gamma$ for BPSK and $SNR = 2\gamma$ for QPSK. This means that for QPSK in Figure 1, the SNR values are 3 dB higher than the γ-values written at the axes. For transmit antenna diversity ($L_r = 1$ and $L_t = 2$), $SNR = 2\gamma$ for BPSK and $SNR = 4\gamma$ for QPSK. This means that for BPSK resp. QPSK in Figure 1, the SNR values are 3 dB resp. 6 dB higher than the γ-values written at the axes.

The validity of the assumption that the loss can be estimated from the right-hand-side of (15) can be checked by comparison with the bit error curves obtained directly from Equations (8) and (9). Figures 2 shows these union bounds for $L_r = 2$, $L_t = 1$ and QPSK with the $R_c = 2/3$ punctured convolutional code obtained from the mother code with generators $(133, 171)_{oct}$ [5]. The black dashed curve corresponds to the union bound (UB) for only one receive antenna. The difference of this curve to the yellow curve corresponding to two totally correlated antennas ($\rho = 1$) is exactly 3.01 dB due to the doubled antenna size at the receiver. This must not be interpreted as a diversity gain, but as an antenna gain. The real diversity gain can be read from the figure as the difference between the yellow curve and the other coloured curves. For ideally uncorrelated antennas ($\rho = 0$), this gain is approximately 2.2 dB at $P_b = 2 \cdot 10^{-4}$. For correlated antennas with, e. g. $\rho = 0.8$, half of this gain (i. e. 1.1 dB) gets lost. For $\rho = 0.6$ the loss is approximately 0.5 dB, and it becomes much smaller for values below $\rho = 0.4$.

To compare these figures with the curves of Figure 1 one must look at the γ_0 =1 dB value at the ordinate. This is because $P_b = 2 \cdot 10^{-4}$ is reached at $SNR = 4$ dB and $\gamma = SNR/2$. From that figure, we read SNR losses of approx. 0.5 dB for $\rho = 0.6$, 1 dB for $\rho = 0.8$, and 2 dB for $\rho = 1$. This is in close agreement to the losses obtained from Figure 2, which justifies our assumption.

We note that although this method has been established only under the prerequisite of ideal interleaving, there is some evidence that it can also be applied if this condition is not fulfilled. This is because one may regard non-ideal interleaving as a weakening of the code that has a similar effect as a decrease of its Hamming distance [6]. Since the formula (18) does not depend on the Hamming distance d but only on the SNR, we expect that it also holds for that case.

IV. An Example For Spatial Correlations

To apply the above method to a concrete scenario, some model assumptions about the spatial correlations have to be introduced. The complex fading amplitude is assumed to be a mean-zero space-dependent random variable. We use a *planar* model in which only the azimuth angle of the incoming (or outgoing) wave at the antennas is relevant for the correlations and all antennas are in one plane. This is reasonable for an essentially planar environment as the surface of the earth. We further assume a sufficiently long distance between transmitter and receiver so that a plane wave model can be applied. We write the spatial distance vector $\mathbf{d}_{ik} = \mathbf{x}_i - \mathbf{x}_k$ between antenna i (located at $\mathbf{x}_i$) and antenna k (located at $\mathbf{x}_k$) in polar coordinates as

$$\mathbf{d}_{ik} = \begin{pmatrix} d_{ik} \cos \beta_{ik} \\ d_{ik} \sin \beta_{ik} \end{pmatrix} , \qquad (19)$$

where d_{ik} is the distance and β_{ik} is the (azimuth) angle between the antenna locations. We assume both antennas as isotropic radiators. In such a model, the correlation coefficient (3) between antenna i and antenna k can be expressed as [7], [8]

$$\rho_{ik} = \int_0^{2\pi} \mathrm{d}\alpha \, \mathrm{e}^{j2\pi \frac{d_{ik}}{\lambda} \cos(\alpha - \beta_{ik})} \mathcal{S}_{\text{angle}} \ (\alpha) \ . \qquad (20)$$

In that equation, λ is the wavelength, and $\mathcal{S}_{\text{angle}} \ (\alpha)$ is the power angular spectrum (PAS), i. e. the angle-dependent power distribution of the signal. This quantity reflects the topographical environment, and a reasonable model for it has to be built either based on measurements or on theoretical assumptions. At the mobile station site, which is typically surrounded by many obstacles (scatterers), the PAS is often modeled as an isotropic distribution of the signal power, i. e.

$$\mathcal{S}_{\text{angle}} \ (\alpha) = \frac{1}{2\pi} \, . \qquad (21)$$

However, other distributions with a moderately anisotropic shape should also be considered. In contrast to the mobile station, the base station site typically towers above its environment, and the PAS is concentrated to a relatively small angle.

For the isotropic scattering PAS (21), the spatial correlation (20) is simply given by

$$\rho_{ik} = \mathrm{J}_0 \left(2\pi \frac{d_{ik}}{\lambda} \right) , \qquad (22)$$

where $\mathrm{J}_0 \, (x)$ is the (ordinary) Bessel function of order zero. If an analytical solution for the integral expression (20) for ρ_{ik} is not available, it may be calculated by numerical quadrature methods. Alternatively, one may express ρ_{ik} by an infinite series and get an approximation by using only a few significant terms of it. To see this, we write the PAS as a Fourier series

$$\mathcal{S}_{\text{angle}} \ (\alpha) = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{jn\alpha} \, . \qquad (23)$$

Such a convergent Fourier series exists for any $\mathcal{S}_{\text{angle}} \ (\alpha)$ that is continuous on the unit circle. In many cases, it is also reasonable to assume that nearly all of the power is contained in a few coefficients c_n with small absolute values of n. We insert Equation (23) into Equation (20), use the integral representation

$$\mathrm{J}_n \, (x) = \frac{1}{2\pi j^n} \int_{-\pi}^{\pi} \mathrm{e}^{jx \cos\phi} \mathrm{e}^{jn\phi} \mathrm{d}\phi \qquad (24)$$

of the n-th order Bessel function and eventually get the series for the correlation coefficient:

$$\rho_{ik} = \sum_{n=-\infty}^{\infty} 2\pi j^n \mathrm{e}^{jn\beta_{ik}} c_n \mathrm{J}_n \left(2\pi \frac{d_{ik}}{\lambda} \right) \, . \qquad (25)$$

Because $\max_x |\mathrm{J}_n \, (x)|$ becomes smaller with increasing absolute value of n, the convergence of the series (25) is even faster than the convergence of the Fourier series (23). If only some coefficients c_n with small absolute values of n are relevant, one can approximately express ρ_{ik} by a finite series with a leading term at $n = 0$ that is proportional to the expression (22) for isotropic scattering and some correction terms proportional to $\mathrm{J}_n \left(2\pi \frac{d_{ik}}{\lambda} \right)$ (for $n \neq 0$) that characterizes the anisotropy.

We discuss the method described above by the example of the wrapped Cauchy distribution which has the following probability density function [9]:

$$\mathcal{S}_{\text{angle}} \ (\alpha) = \frac{1}{2\pi} \frac{1 - a^2}{1 + a^2 - 2a \cos\alpha} \qquad (26)$$

The parameter a with $0 < a < 1$ controls the opening angle of the PAS. For $a \to 0$, the PAS becomes more and more isotropic. The Fourier coefficients can be expressed analytically [9] by:

$$c_n = \frac{a^{|n|}}{2\pi} . \qquad (27)$$

The upper part of Figure 3 shows the PAS defined by Equation (26) for different parameters of a. For small values of a, this may serve as a model for the mobile station and for higher values of a as a model for the base station. The lower part of the figure shows the corresponding correlation coefficient $|\rho_{12}|$

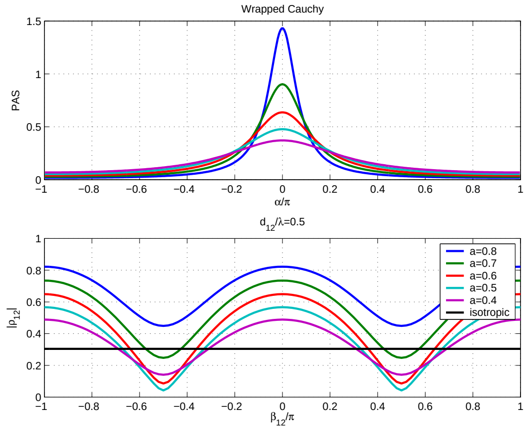

Fig. 3. The PAS $\mathcal{S}_{\text{angle}}\ (\alpha)$ for the wrapped Cauchy distribution (upper figure) and the corresponding correlation coefficient $|\rho_{12}|$ in dependence of the angle β_{12} for fixed distance $d_{12} = \lambda/2$ (lower figure).

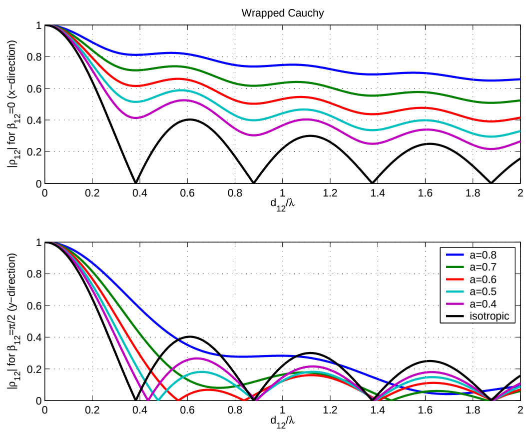

Fig. 4. The correlation coefficient $|\rho_{12}|$ for the wrapped Cauchy distribution in dependence of the distance $d_{12} = \lambda/2$ for $\beta_{12} = 0$ (x-direction, upper figure) and for $\beta_{12} = \pi/2$ (y-direction, lower figure).

calculated by the series (25) for a fixed distance $d_{12} = \lambda/2$ between the antennas as a function of the angle β_{12}, which is the angle between the vector $\mathbf{d}_{12}$ and the direction of the maximal power (maximum of $\mathcal{S}_{\text{angle}}\ (\alpha)$). One can see from this figure that the correlation is maximal if $\mathbf{d}_{12}$ is parallel to that direction ($\beta_{12} = 0$ or $\beta_{12} = \pi$), and it is minimal if $\mathbf{d}_{12}$ is perpendicular ($\beta_{12} = \pm\pi/2$). For these extrema, the dependence of $|\rho_{12}|$ on the distance is plotted in Figure 4 . The correlation decays very slowly if both antennas are in line with the angle of maximal power (upper figure). This is the worst case that must be taken into account. If a correlation coefficient $|\rho_{12}| \lesssim 0.4$ is requested (cf. our discussion in the preceding section), an antenna separation of several wavelengths is necessary for $a \gtrsim 0.6$. Such values of a are likely at the base station site. For the isotropic scattering case (black curve), which may be a reasonable assumption at the mobile station site, the popular rule-of-thumb $d_{12} \gtrsim \lambda/2$ is justified because it leads to $|\rho_{12}| \lesssim 0.4$. For a moderately anisotropic scattering with $a \approx 0.4$, an antenna separation $d_{12} \approx \lambda$ is sufficient.

V. Conclusions

In this paper, we have presented a formula that allows a simple estimate of the SNR loss due to spatial correlations between two antennas in a diversity system. In Figure 1, we have shown a set of curves from which this loss can be read as a function of the SNR with the correlation coefficient as a parameter. Finally, in our example we have demonstrated how this correlation coefficient can be calculated numerically from a series expansion that is available for any PAS with known Fourier coefficients.

References

[1] J. Proakis and M. Salehi, *Digital Communications*, 5th ed. McGraw-Hill, 2008.

[2] S. Alamouti, “A simple transmit diversity technique for wireless communications,” *IEEE J. Select. Areas Comm.*, vol. 16, pp. 1452–1458, 1998.

[3] H. Schulze and C. Lüders, *Theory and Practice of OFDM and CDMA - Wideband Wireless Communications*. Wiley, 2005.

[4] H. Schulze, “Concatenated space-time coding with two transmit antennas,” in *Proc. 4th ITG Conference on Source and Channel Coding Berlin*, January 2002, pp. 407–414.

[5] J. Hagenauer, “Rate-compatible punctured convolutional codes (RCPC codes) and their applications,” *IEEE Trans. Commun.*, vol. 36, pp. 389–400, April 1988.

[6] H. Schulze, “The performance of multicarrier CDMA for the correlated Rayleigh fading channel,” *AEÜ International Journal of Electronics and Communications*, vol. 55(2), pp. 88–94, 2001.

[7] W. C. Y. Lee, “Effects of correlation between two mobile radio base-station antennas.” *IEEE Trans. Commun.*, vol. 21(11), pp. 1214–1223, November 1973.

[8] W. C. Jakes, *Microwave Mobile Communications*. Wiley-Interscience, 1974.

[9] P. Ravindran, “Bayesian analysis of circular data using wrapped distributions,” Ph.D. dissertation, North Carolina State University, 2002.